New Advances in Nonparametric IRT

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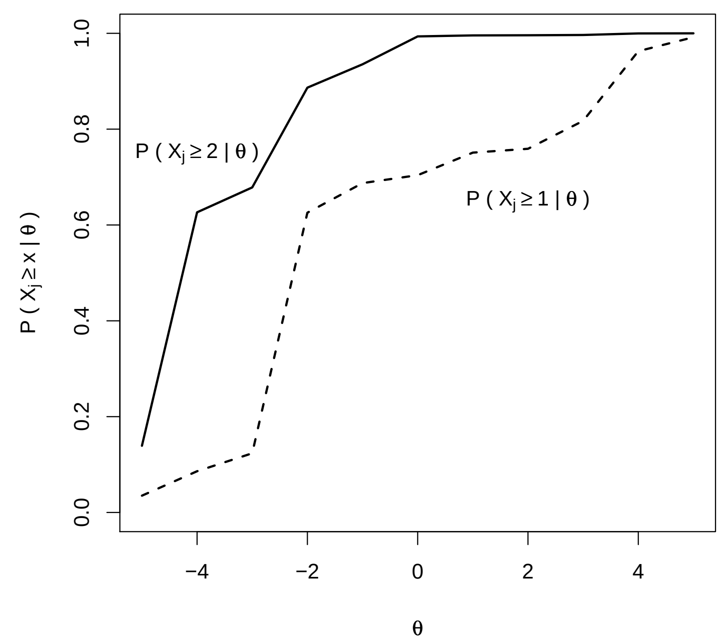
Wicher Bergsma, Marcel Croon, Rudy Ligtvoet, Irena Mikolajun, Klaas Sijtsma, Hendrik Straat, Joost van Ginkel, Wobbe Zijlstra

NIRT: Class of IRT models consisting of weak nonparametric assumptions e.g., Junker & Sijtsma (2000); Holland & Rosenbaum (1986); Mokken (1971); Ramsay (1991); Sijtsma & Molenaar (2003); Stout (1990)

Example: Monotone Homogeneity Model (MHM Mokken, 1971; Molenaar, 1997)

- Unidimensionality
- Local independence
- Monotonicity:
  $P(X_j \geq x|\theta)$ nondecreasing in θ



Ordinal measurement using NIRT models:

- Dichotomous items: MHM implies SO of θ by X_+

  e.g., Grayson (1988)

$$E(\theta|X_+ = x - 1) \leq E(\theta|X_+ = x)$$

- Polytomous items: MHM implies weak SO of θ by X_+

  Van der Ark & Bergsma (submitted)

$$E(\theta|X_+ < x) \leq E(\theta|X_+ \geq x)$$

NIRT and real data:

1. Investigate fit: **Mokken Scale Analysis**

Logic:	NIRT model	$\Rightarrow$	Observable property
	Observable property failed	$\Rightarrow$	Reject NIRT model
Example	MHM	$\Rightarrow$	all $\text{Cov}(X_i, X_j) \geq 0$
	Some $\text{Cov}(X_i, X_j) < 0$	$\Rightarrow$	reject MHM

   (a) Partitioning items into *Mokken scales*
   (b) Investigating model assumptions per scale

   Note: Parametric IRT models $\Rightarrow$ NIRT models

2. Interpret NIRT model: Estimate IRF, information on item scalability, etc.

Advances in NIRT

Improvement of item selection algorithm (Straat, Friday)

Ordering of items (Ligtvoet, Friday)

Outlier detection (Zijlstra, Friday)

Testing observable properties using categorical marginal models

Van der Ark, Croon, & Sijtsma (2008); Bergsma, Croon, & Hagenaars (2009)

Scalability coefficients

$$H_{ij} = \frac{\mathsf{Cov}(X_i,X_j)}{\mathsf{Cov}^{\mathsf{max}}(X_i,X_j)} \qquad \text{Item pair}$$

$$H_i = \frac{\sum_j \mathsf{Cov}(X_i,X_j)}{\sum_j \mathsf{Cov}^{\mathsf{max}}(X_i,X_j)} \qquad \text{Item}$$

$$H = \frac{\sum_i \sum_j \mathsf{Cov}(X_i,X_j)}{\sum_i \sum_j \mathsf{Cov}^{\mathsf{max}}(X_i,X_j)} \qquad \text{Scale} \qquad \begin{cases} H > .30 & \text{Weak scale} \\ H > .40 & \text{Moderate scale} \\ H > .50 & \text{Strong scale} \end{cases}$$

Mokken scale criteria

Criterion 1: $H_{ij} > 0$
Criterion 2: $H_i > c \geq 0$

Small Real-Data Example: Three balance tasks

Data

	X_1	X_2	X_3
1	1	1	0
2	1	1	0
3	0	0	0
4	1	1	1
5	1	1	1
6	1	0	0
7	0	0	0
8	1	1	1
9	0	0	0
10	1	1	0
⋮	⋮	⋮	⋮
484	1	1	1

Coefficients

H_{ij}			
	X_1	X_2	X_3
X_1	1.00	.45	.62
X_2	.45	1.00	.55
X_3	.62	.55	1.00
	X_1	X_2	X_3
H_i	.51	.49	.58
P	.84	.79	.44
H	.53		
alpha	.54		

Challenges:

- Standard errors H_i
- Standard errors H_{ij}, H_i, and H for polytomous items
- Simultaneous testing: $\mathbf{H}_{ij} > \mathbf{0}$ (Criterion 1)
- Simultaneous testing: $\mathbf{H}_{ij} > \mathbf{d}$ (Stronger Criterion 1)
- Simultaneous testing: $\mathbf{H}_i > \mathbf{c}$ (Criterion 2)
- Testing scale: $H = .50$
- Testing equal item-scalability coefficients
- Comparing groups: $H^1 = H^2 = \ldots = H^g$

Categorical Marginal models (CMMs)

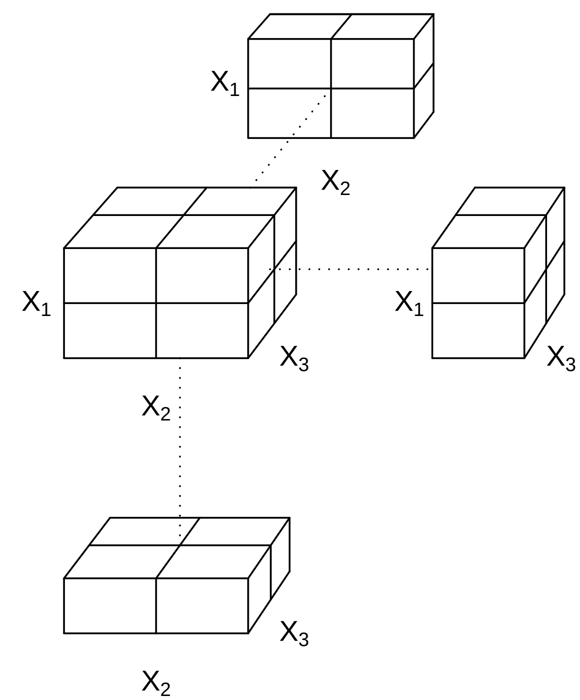
- Collect data in contingency-table: $\mathbf{n}$

	X_1	X_2	X_3
1	1	1	0
2	1	1	0
3	0	0	0
4	1	1	1
5	1	1	1
6	1	0	0
7	0	0	0
8	1	1	1
9	0	0	0
10	1	1	0
⋮	⋮	⋮	⋮
484	1	1	1

$$\mathbf{n} = \begin{array}{cc} \begin{pmatrix} 42 \\ 3 \\ 24 \\ 10 \\ 40 \\ 17 \\ 167 \\ 181 \end{pmatrix} & \begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} \end{array}$$

Categorical Marginal models (CMMs)

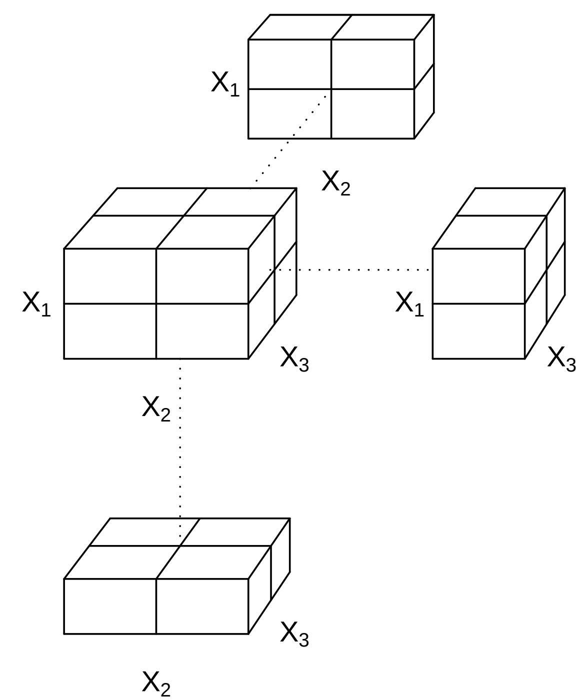
- Collect data in contingency-table: $\mathbf{n}$
- CMMs estimate frequencies $\hat{\mathbf{m}}$ under constraints in (functions of) marginals



Categorical Marginal models (CMMs)

- Collect data in contingency-table: $\mathbf{n}$
- CMMs estimate frequencies $\hat{\mathbf{m}}$ under constraints in (functions of) marginals
- Examples of constraints

  $H_{12} = H_{13} = H_{23}$



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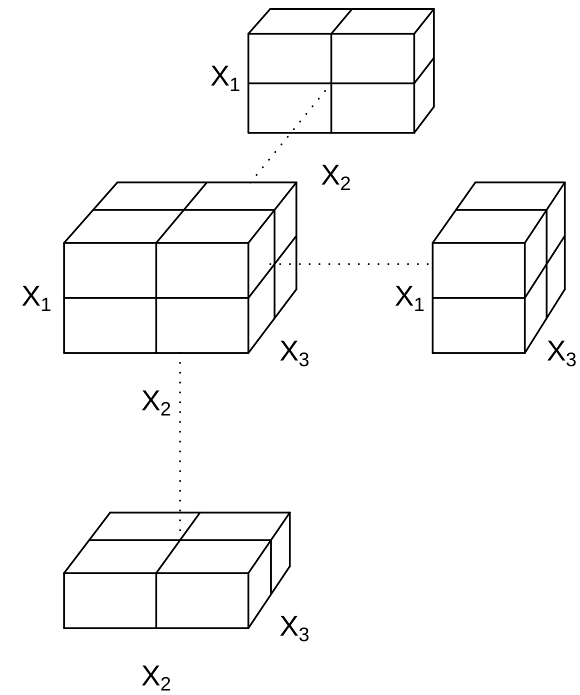

$$H_{12} = \frac{\mathrm{Cov}(X_1, X_2)}{\mathrm{Cov}^{\max}(X_1, X_2)}$$

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- Collect data in contingency-table: $\mathbf{n}$
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  $H_{12} = H_{13} = H_{23}$

	Obs.	Exp.
Coefficients		
H_{12}	.45	.51
H_{13}	.62	.51
H_{23}	.55	.51
Frequencies		
000	42	42.2
001	3	5.6
010	24	19.2
011	10	11.6
100	40	39.4
101	17	17.2
110	167	171.1
111	181	177.6
Global fit		
G^2	2.99	
df	2	

Categorical Marginal models (CMMs)

- Collect data in contingency-table: $\mathbf{n}$
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- Examples of constraints

$H_{12} = H_{13} = H_{23}$

$H = .50$

	Obs.	Exp.
Coefficients		
H	.53	.50
Frequencies		
000	42	40.3
001	3	3.2
010	24	24.7
011	10	10.8
100	40	39.8
101	17	17.8
110	167	167.7
111	181	179.6
Global fit		
G^2	0.23	
df	1	

Categorical Marginal models (CMMs)

- Collect data in contingency-table: $\mathbf{n}$
- CMMs estimate frequencies $\hat{\mathbf{m}}$ under constraints in (functions of) marginals
- Examples of constraints

  $H_{12} = H_{13} = H_{23}$

  $H = .50$

  $(H_1, H_2, H_3) = (.3, .3, .3)$

	Obs.	Exp.
Coefficients		
H_1	.51	.30
H_2	.49	.30
H_3	.58	.30
Frequencies		
000	42	28.9
001	3	5.7
010	24	25.2
011	10	18.9
100	40	37.6
101	17	24.6
110	167	176.0
111	181	167.1
Global fit		
G^2	16.23	
df	3	

Categorical Marginal models (CMMs)

- Collect data in contingency-table: $\mathbf{n}$
- CMMs estimate frequencies $\hat{\mathbf{m}}$ under constraints in (functions of) marginals
- Examples of constraints

  $H_{12} = H_{13} = H_{23}$

  $H = .50$

  $(H_1, H_2, H_3)^T = \mathbf{0}$
- Beyond Mokken scale analysis

  Cronbach's alpha = .70

  Testing many other coefficients

  Marginal homogeneity:

  $P(X_1 = 1) = P(X_2 = 1) = P(X_3 = 1)$

  Loglinear modelling

  Panel-data analysis.

	Obs.	Exp.
Coefficients		
α	.54	.70
Frequencies		
000	42	72.7
001	3	1.9
010	24	19.8
011	10	6.3
100	40	34.3
101	17	11.1
110	167	141.8
111	181	196.2
Global fit		
G^2	27.65	
df	1	

ML estimation

$$
\begin{aligned}
L(\mathbf{m}, \boldsymbol{\lambda}) &= \mathbf{n}^T \log(\mathbf{m}) - \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{m}) \\
l(\mathbf{m}, \boldsymbol{\lambda}) &= \mathbf{n} - \mathbf{m} - \mathbf{G}(\mathbf{m}) \boldsymbol{\lambda}^T
\end{aligned}
$$

$\mathbf{G}(\mathbf{m})$, Jacobian of $\mathbf{g}(\mathbf{m})$, may be difficult to obtain unless constraints expressed in a *general exp-log notation*.

$$
\mathbf{g}(\mathbf{m}) = \phi_q(\mathbf{A}_q \phi_{q-1}(\mathbf{A}_{q-1} \ldots \phi_2(\mathbf{A}_2 \phi_1(\mathbf{A}_1 \mathbf{m})))) = \mathbf{0}
$$

$$
\mathbf{A}_i : \text{ Design matrix; } \quad
\begin{aligned}
\phi_i(\mathbf{X}) &: \exp(\mathbf{X}) \\
&: \log(\mathbf{X}) \\
&: \mathbf{X} + c
\end{aligned}
$$

ML estimation

For example: $H_{12} = \frac{\text{Cov}(X_1,X_2)}{\text{Cov}^{\max}(X_1,X_2)} = 1 - \frac{m_{+++}m_{01+}}{m_{0++}m_{+1+}} = .5$

$$.5 - \exp\left[\mathbf{A_2} \log\left[\mathbf{A_1 m}\right]\right] = 0$$

$$.5 - \exp\left[\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \log\left[\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} m_{000} \\ m_{001} \\ m_{010} \\ m_{011} \\ m_{100} \\ m_{101} \\ m_{110} \\ m_{111} \end{pmatrix}\right]\right] = 0$$

$$.5 - \exp\left[\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \log\begin{pmatrix} m_{+++} \\ m_{0++} \\ m_{+1+} \\ m_{01+} \end{pmatrix}\right] = 0$$

$$.5 - \frac{m_{+++}m_{01+}}{m_{0++}m_{+1+}} = 0$$

ML estimation

Constraint

$$H = c$$

Constraint in generalized exp-log notation

$$g(\mathbf{m}) = 1 - \exp(\mathbf{A}_4 \log(\mathbf{A}_3 \exp(\mathbf{A}_2 \log(\mathbf{A}_1 \mathbf{m})))) - c = 0.$$

With

$$\mathbf{A}_1 = \begin{pmatrix} \mathbf{1}_q^T \\ \mathbf{1}\mathbf{1}^T - \mathbf{R}^T \\ \mathbf{R}^T \\ (\mathbf{1}\mathbf{1}^T - \mathbf{R}^T) \circledast \mathbf{R}^T \end{pmatrix} \quad \mathbf{A}_2 = 1 \oplus \mathbf{Q}_1 \oplus \mathbf{I}_L$$

$$\mathbf{A}_3 = 1 \oplus \mathbf{Q}_2 \oplus \mathbf{Q}_2 \qquad \mathbf{A}_4 = \begin{pmatrix} \mathbf{1}_J & -\mathbf{I}_J & \mathbf{I}_J \end{pmatrix}.$$

Asymptotic standard errors

$$\mathbf{\Sigma}(\mathbf{g}(\hat{\mathbf{m}})) = \mathbf{G}(\hat{\mathbf{m}})^T \mathbf{\Sigma}(\hat{\mathbf{m}}) \mathbf{G}(\hat{\mathbf{m}})$$

Coefficients

H_{ij}	X_1	X_2	X_3
X_1	1.00	.45	.62
X_2	.45	1.00	.55
X_3	.62	.55	1.00

	X_1	X_2	X_3
H_i	.51	.49	.58
P	.84	.79	.44
H	.53		
alpha	.54		

Coefficients

H_{ij}	X_1	X_2	X_3
X_1	1.00	.45	.62
		(.07)	(.09)
X_2	.45	1.00	.55
	(.07)		(.08)
X_3	.62	.55	1.00
	(.09)	(.08)	

	X_1	X_2	X_3
H_i	.51	.49	.58
	(.06)	(.06)	(.07)
P	.84	.79	.44
	(.02)	(.02)	(.02)
H	.53		
	(.05)		
alpha	.54		
	(.04)		

Considerations

- Size of design matrices rapidly increase as number of items increases

- Equality and inequality constraints

- Assumption: Observed item ordering (P-values) is true ordering.

- Sparse contingency tables not problematic.

- CMMs can be applied to all types of coefficient

Software (R-packages)

Van der Ark, L. A. (2008). `mokken`: Mokken scale analysis in R.

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